AP CALC Summer Packet

2023

Tilton School

Upcoming AP Calculus AB Students,

Congratulations on committing to this rigorous course and beginning your introduction to the study of Calculus! This class will draw upon a large amount of prior knowledge from your previous classes, like Algebra, Geometry, Trigonometry, and Pre-Calculus, which is why completing this summer packet is highly beneficial to your success throughout this year. Considering the high level of this course and the rigorous pacing of Advanced Placement curriculum, content from this packet and your previous math courses are expected to be understood and will not be retaught within class time.

The following packet was designed to help you remember the very basics of what you are supposed to know. This represents the very minimum of what you should complete over the summer. If you have difficulty on some of the subjects, try these websites to watch videos and tutorials as well as getting some practice problems with answers and immediate feedback.

1) Hippo Campus: http://www.hippocampus.org/
2) Khan Academy: https://khanacademy.org/
3) Wolfram Alpha: http://www.wolframalpa.org/

Of course you can use other options to refresh your learned knowledge but it is up to you to know, retain, and be able to use the prerequisite knowledge from previous years.

The following pages contain the functions you need to know, some general terms, formulas, or concepts, as well as practice problems. Please work on this packet throughout the next few weeks individually, with other classmates, or with my own help once you have returned to school. Once again, I will not reteach these concepts during class time, however, students are more than welcome to review concepts with myself outside of class and I encourage you to do so at the earliest moment in which you feel you need help!

This packet is to be completed by the first day of class,

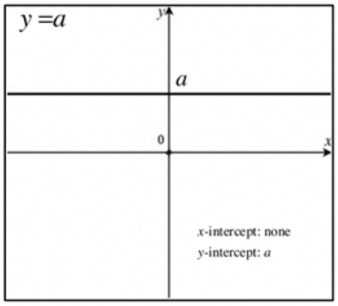
Wednesday September 6th, 2023

I look forward to working with you all soon!

Mrs. Compton
rcompton@tiltonschool.org

GRAPHS OF COMMON FUNCTIONS

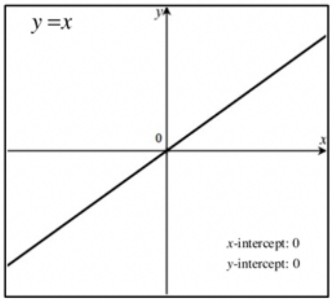
There are certain graphs that occur all the time in calculus and students should know the general shape of them, where they hit the x-axis (zeros) and y-axis (y-intercept), as well as the domain and range. There are no assignment problems for this section other than students memorizing the shape of all of these functions. In section 5, we will talk about transforming these graphs.



Function: $y = a$

Domain: $(-\infty, \infty)$

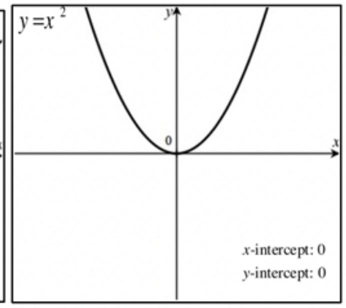
Range: $[a, a]$



Function: $y = x$

Domain: $(-\infty, \infty)$

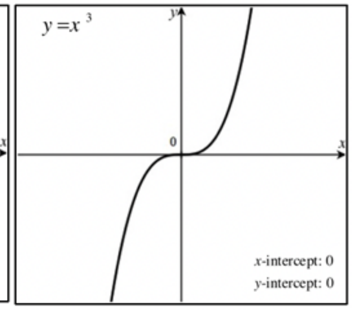
Range: $(-\infty, \infty)$



Function: $y = x^2$

Domain: $(-\infty, \infty)$

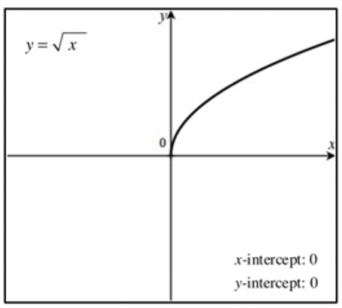
Range: $[0, \infty)$



Function: $y = x^3$

Domain: $(-\infty, \infty)$

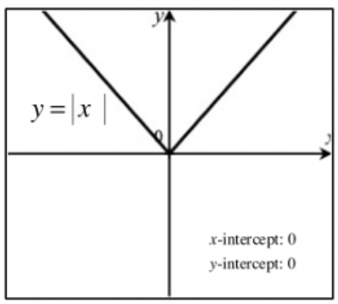
Range: $(-\infty, \infty)$



Function: $y = \sqrt{x}$

Domain: $[0, \infty)$

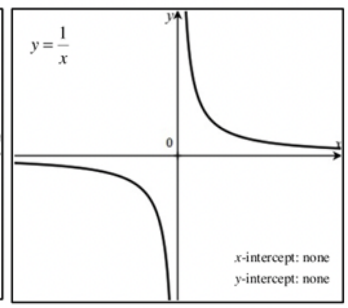
Range: $[0, \infty)$



Function: $y = |x|$

Domain: $(-\infty, \infty)$

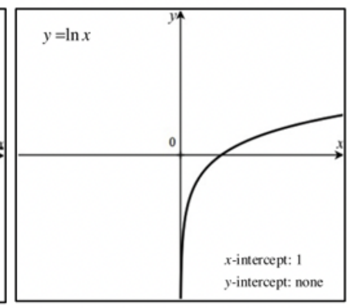
Range: $[0, \infty)$



Function: $y = \dfrac{1}{x}$

Domain: $x \neq 0$

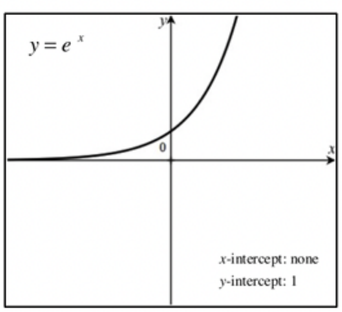
Range: $y \neq 0$



Function: $y = \ln x$

Domain: $(0, \infty)$

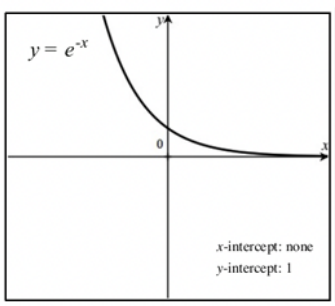
Range: $(-\infty, \infty)$



Function: $y = e^x$

Domain: $(-\infty, \infty)$

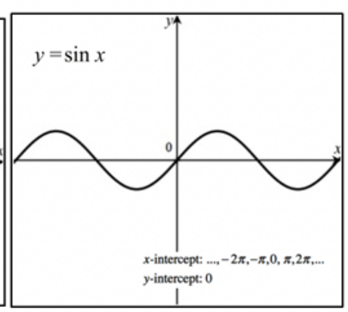
Range: $(0, \infty)$



Function: $y = e^{-x}$

Domain: $(-\infty, \infty)$

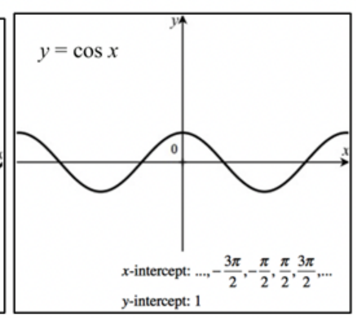
Range: $(0, \infty)$



Function: $y = \sin x$

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$



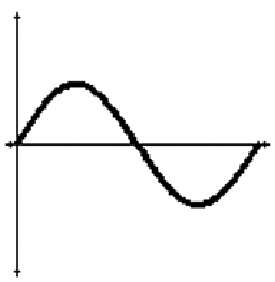
Function: $y = \cos x$

Domain: $(-\infty, \infty)$

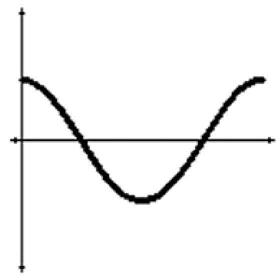
Range: $[-1, 1]$

Trig Functions

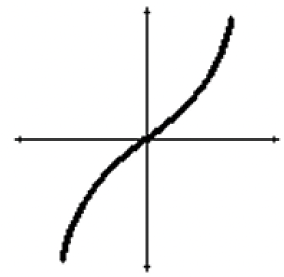
$f(x) = \sin x$



$f(x) = \cos x$



$f(x) = \tan x$



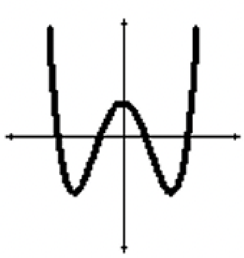
Polynomial Functions:

A function P is called a polynomial if $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$
Where n is a nonnegative integer and the numbers a_0, a_1, a_2, … a_n are constants.

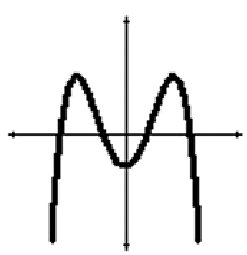
Even degree

Leading coefficient sign

Positive



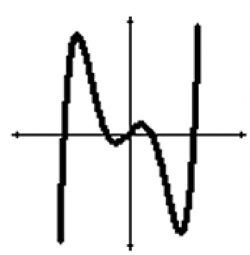
Negative



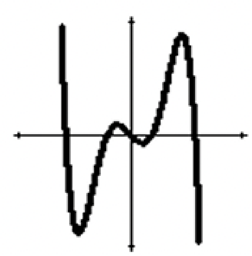
Odd degree

Leading coefficient sign

Positive



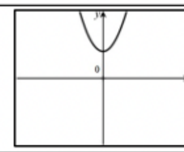
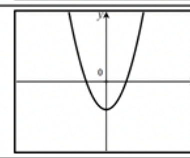
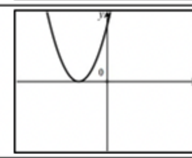
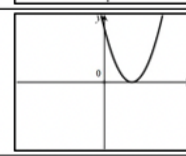
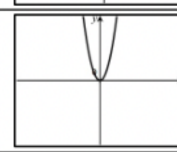
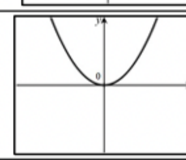
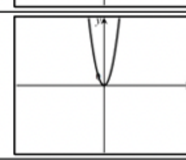
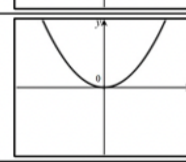
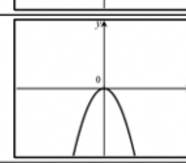
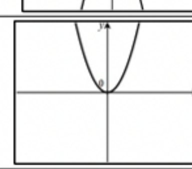
Negative



- Number of roots equals the degree of the polynomial.
- Number of x intercepts is less than or equal to the degree.
- Number of “turns” is less than or equal to (degree – 1).

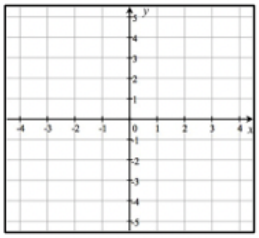
TRANSFORMATIONS OF FUNCTIONS

A curve in the form $y = f(x)$, which is one of the basic common functions from section C can be transformed in a variety of ways. The shape of the resulting curve stays the same but zeros and y-intercepts might change and the graph could be reversed. The table below describes transformations to a general function $y = f(x)$ with the parabolic function $f(x) = x^2$ as an example.

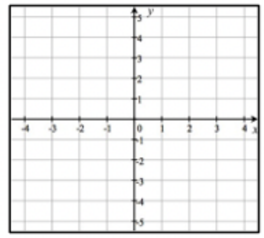
Notation	How $f(x)$ changes	Example with $f(x) = x^2$
$f(x)+a$	Moves graph up a units	
$f(x)-a$	Moves graph down a units	
$f(x+a)$	Moves graph a units left	
$f(x-a)$	Moves graph a units right	
$a \cdot f(x)$	$a > 1$: Vertical Stretch	
$a \cdot f(x)$	$0 < a < 1$: Vertical shrink	
$f(ax)$	$a > 1$: Horizontal compress (same effect as vertical stretch)	
$f(ax)$	$0 < a < 1$: Horizontal elongated (same effect as vertical shrink)	
$-f(x)$	Reflection across x-axis	
$f(-x)$	Reflection across y-axis	

• Sketch the following equations:

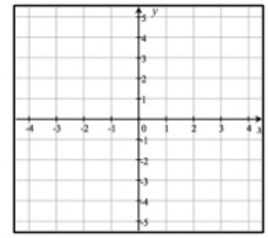
1. $y=-x^2$



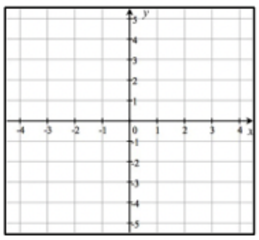
2. $y=2x^2$



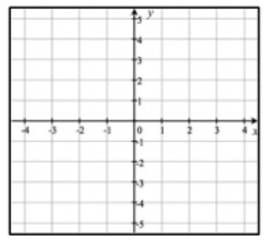
3. $y=(x-2)^2$



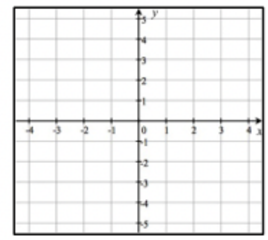
4. $y=2-\sqrt{x}$



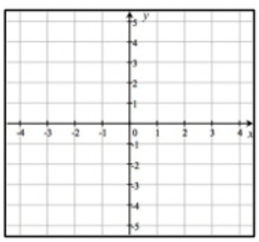
5. $y=\sqrt{x+1}+1$



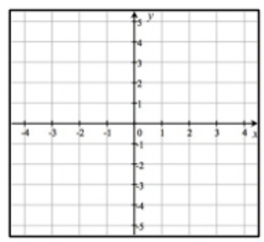
6. $y=\sqrt{4x}$



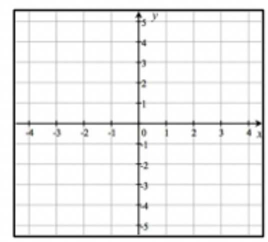
7. $y=|x+1|-3$



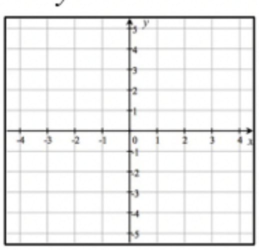
8. $y=-2|x-1|+4$



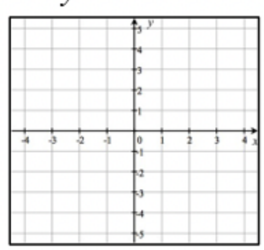
9. $y=-\left|\frac{x}{2}\right|-1$



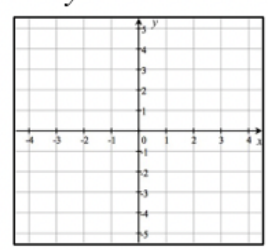
10. $y=2^x-2$



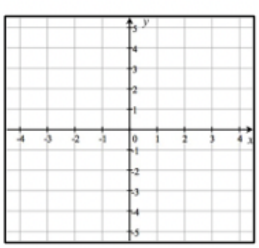
11. $y=-2^{x+2}$



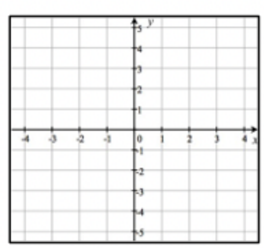
12. $y=2^{-2x}$



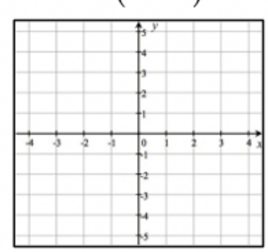
13. $y=\frac{1}{x-2}$



14. $y=\frac{-2}{x+1}$



15. $y=\frac{1}{(x+2)^2}-3$



FUNCTION NOTATION & EVALUATING FUNCTIONS

To evaluate a function for a given value, simply plug the value into the function for x.

Recall: $(f \circ g)(x) = f(g(x))$ *OR* $f[g(x)]$ read "***f*** of ***g*** of ***x***" Means to plug the inside function (in this case g(x)) in for x in the outside function (in this case, f(x)).

Example: Given $f(x) = 2x^2 + 1$ *and* $g(x) = x - 4$ find *f(g(x))*.

$$\begin{aligned} f(g(x)) &= f(x-4) \\ &= 2(x-4)^2 + 1 \\ &= 2(x^2 - 8x + 16) + 1 \\ &= 2x^2 - 16x + 32 + 1 \\ f(g(x)) &= 2x^2 - 16x + 33 \end{aligned}$$

Let $f(x) = 2x + 1$ **and** $g(x) = 2x^2 - 1$**. Find each:**

$f(2) =$ ________ $g(-3) =$ ________ $f(t + 1) =$ ________

$f(g(-2)) =$ ________ $g(f(m + 2)) =$ ________ $[f(x)]^2 - 2g(x) =$ ________

Let $f(x) = sin(2x)$**. Find each exactly.**

$f(\frac{\pi}{4}) =$ ________ $f(\frac{2\pi}{3}) =$ ________

Let $f(x) = x^2,\ g(x) = 2x + 5,\ and\ h(x) = x^2 - 1$**. Find each**

$h(f(-2)) =$ ________ $f(g(x - 1)) =$ ________ $g(h(x^3)) =$ ________

EQUATION OF A LINE

Slope intercept form: $y = mx + b$

Vertical line: x = c (slope is undefined)

Point-slope form: $y - y_1 = m(x - x_1)$
* LEARN! We will use this formula frequently!

Horizontal line: y = c (slope is 0)

Example: Write a linear equation that has a slope of ½ and passes through the point (2, -6)

Slope intercept form

$y = \frac{1}{2}x + b$	Plug in ½ for m
$-6 = \frac{1}{2}(2) + b$	Plug in the given ordered
$b = -7$	Solve for b
$y = \frac{1}{2}x - 7$	

Point-slope form

$y + 6 = \frac{1}{2}(x - 2)$	Plug in all variables
$y = \frac{1}{2}x - 7$	Solve for y

Determine the equation of a line passing through the point (5, -3) with an undefined slope.

Determine the equation of a line passing through the point (-4, 2) with a slope of 0

Use point-slope form to find the equation of the line passing through the point (0, 5) with a slope of $\frac{2}{3}$

Use point-slope form to find a line passing through the point (2, 8) and parallel to the line $y = \frac{5}{6}x - 1$

Use point-slope form to find a line perpendicular to $y =- 2x + 9$ **passing through the point (4, 7)**

Find the equation of a line passing through the points (-3, 6) and (1, 2)

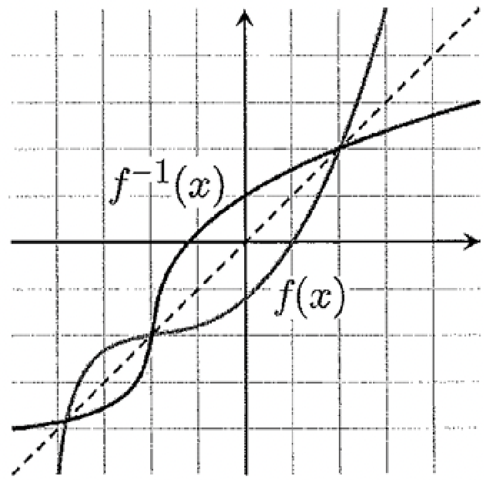
Find the equation of a line with an x-intercept (2, 0) and a y-intercept (0, 3)

INVERSES

To find the inverse of a function, simply switch the x and the y and solve for the new "y" value.
Recall $f^{-1}(x)$ is defined as the inverse of $f(x)$

Example 1:

$f(x)=\sqrt[3]{x+1}$	Rewrite f(x) as y
$y=\sqrt[3]{x+1}$	Switch x and y
$x=\sqrt[3]{y+1}$	Solve for your new y
$(x)^3=\left(\sqrt[3]{y+1}\right)^3$	Cube both sides
$x^3=y+1$	Simplify
$y=x^3-1$	Solve for y
$f^{-1}(x)=x^3-1$	Rewrite in inverse notation



Find the inverse, $f^{-1}(x)$, for each function

$f(x) = 2x + 1$

$f(x) = \frac{x^2}{3}$

$f(x) = \frac{5}{x-2}$

$f(x) = \sqrt{4 - x} + 1$

If the graph of f(x) has the point (2, 7), then what is one point that will be on the graph of $f^{-1}(x)$?

Explain how the graphs of $f(x)$ and $f^{-1}(x)$ compare.

SPECIAL FACTORIZATION

Common factor: $x^3 + x^2 + x = x\left(x^2 + x + 1\right)$

Difference of squares: $x^2 - y^2 = (x + y)(x - y)$ or $x^{2n} - y^{2n} = \left(x^n + y^n\right)\left(x^n - y^n\right)$

Perfect squares: $x^2 + 2xy + y^2 = (x + y)^2$

Perfect squares: $x^2 - 2xy + y^2 = (x - y)^2$

Sum of cubes: $x^3 + y^3 = (x + y)\left(x^2 - xy + y^2\right)$ - Trinomial unfactorable

Difference of cubes: $x^3 - y^3 = (x - y)\left(x^2 + xy + y^2\right)$ - Trinomial unfactorable

Grouping: $xy + xb + ay + ab = x(y + b) + a(y + b) = (x + a)(y + b)$

$4a^2 + 2a$

$x^2 + 16x + 64$

$4x^2 - 64$

$5x^4 - 5y^4$

$16x^2 - 8x + 1$

$9a^4 - a^2b^2$

$2x^2 - 40x + 200$

$x^3 - 8$

$8x^3 + 27y^3$

$x^4 + 11x^2 - 80$

$x^4 - 10x^2 + 9$

$36x^2 - 64$

$x^3 - x^2 + 3x - 3$

$x^3 + 5x^2 - 4x - 20$

$9 - (x^2 + 2xy + y^2)$

SOLVING QUADRATICS

Solving quadratics often requires factoring, but if not factorable, requires the quadratic formula. To factor a quadratic,

Ex:

- Quadratics factor into two binomials
- First terms need to multiple to the first term in the quadratic
- Last terms need to multiple to last term in quadratic
- The correct combination of those values will combine to the middle term
- Solve the quadratic by setting each binomial equal to zero and solving

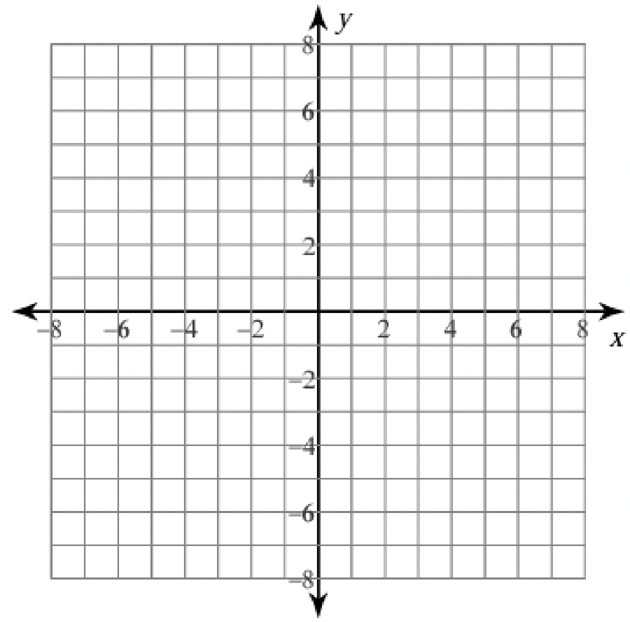
If a quadratic is not factorable, use the quadratic formula

$x^2 + x - 12$	$x^2 - 12x + 35$	$x^2 - 3x - 18$
$x^2 + 2x - 24$	$x^2 - 24x + 132$	$x^2 - 21x - 110$
$30x^2 + 5x - 10$	$9x^2 + 12x + 4$	$3x^2 - 9x - 12$
$27x^2 - 3$	$2x^2 - 5x + 1$	$15x^2 + 8x - 16$

PIECEWISE FUNCTIONS

Graph the piecewise function given below, then evaluate the function at the given values

$$f(x) = \begin{cases} x+5 & x<-2 \\ -2x-1 & x\geq -1 \end{cases}$$

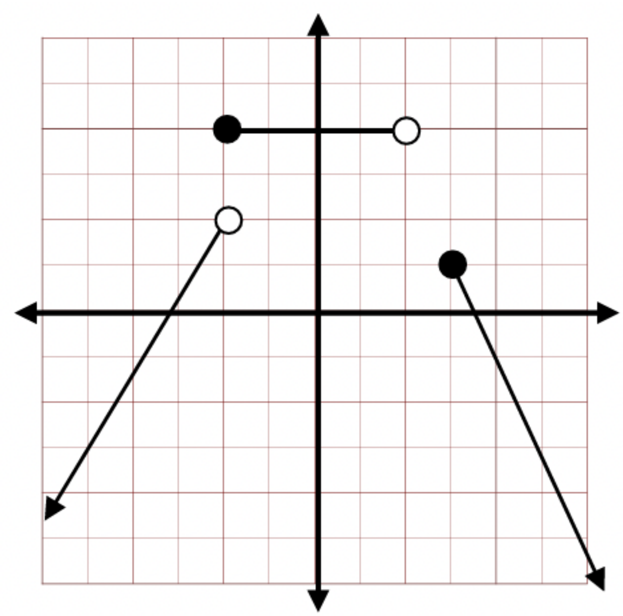


$f(3) =$

$f(-4) =$

$f(-2) =$

Write an equation for the piecewise graph shown



GRAPHING RATIONAL FUNCTIONS

To Identify Types of Discontinuity:

Step 1: **HOLES (Removable Discontinuities)**

- ✓ Factor numerator & denominator
- ✓ Simplify
- ✓ If anything cancels, then there is a hole (More than one factor cancels → More than one hole)
- ✓ Find the ordered pair, (x, y), substitute *x* into the ***SIMPLIFIED EQUATION*** to get *y*

Step 2: **VERTICAL ASYMPTOTES** ***(USE SIMPLIFIED EQUATION)***

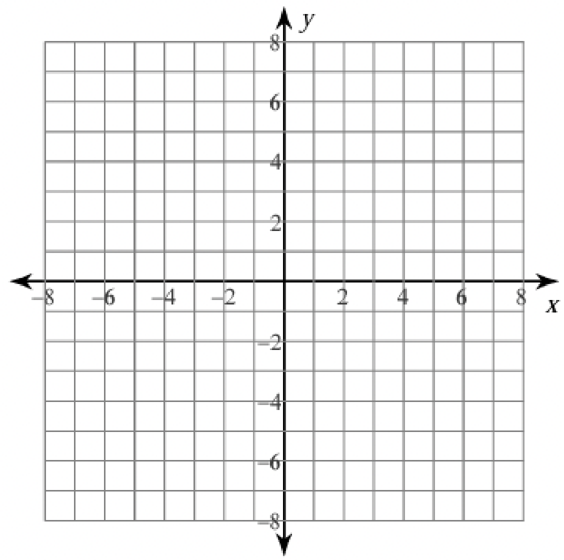
- ✓ Set simplified equation denominator = 0, solve for *x*

Step 3: **HORIZONTAL ASYMPTOTES – *Two Cases (USE SIMPLIFIED EQUATION)***

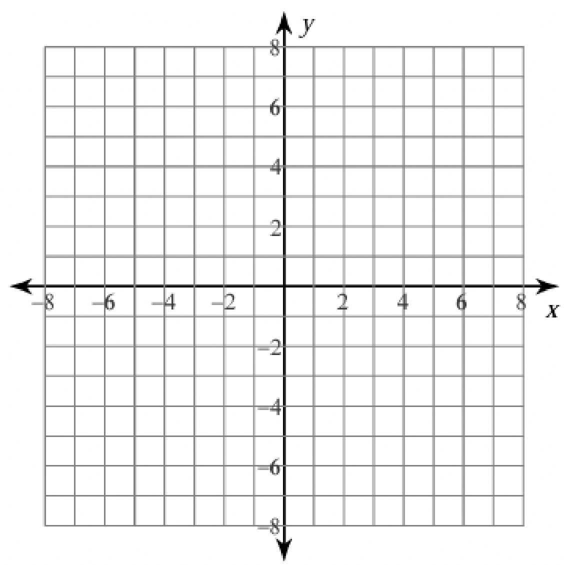
- ✓ Degree of Denominator = Degree of Numerator → $y =$ ratio of leading coefficients
- ✓ Degree of Denominator > Degree of Numerator → $y = 0$

State each discontinuity and sketch the graph of the following rational functions

$g(x) = \frac{4}{x^2 - 3x}$



$f(x) = \frac{2x^2 + 10x + 12}{x^2 + 3x + 2}$



EXPONENT PROPERTIES

In calculus, you will be required to perform algebraic manipulations with **negative exponents** as well as **fractional exponents**. You should know the definition of a negative exponent: $x^{-n} = \frac{1}{x^n}, x \neq 0$. Note that negative powers do not make expressions negative; they create fractions. Typically expressions in multiple-choice answers are written with positive exponents and students are required to eliminate negative exponents. Fractional exponents create roots. The definition of $x^{1/2} = \sqrt{x}$ and $x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$.

As a reminder: when we multiply, we add exponents: $\left(x^a\right)\left(x^b\right) = x^{a+b}$.

When we divide, we subtract exponents: $\frac{x^a}{x^b} = x^{a-b}, x \neq 0$

When we raise powers, we multiply exponents: $\left(x^a\right)^b = x^{ab}$

In your algebra course, leaving an answer with a radical in the denominator was probably not allowed. You had to rationalize the denominator: $\frac{1}{\sqrt{x}}$ changed to $\left(\frac{1}{\sqrt{x}}\right)\left(\frac{\sqrt{x}}{\sqrt{x}}\right) = \frac{\sqrt{x}}{x}$. In calculus, you will find that it is not necessary to rationalize and it is recommended that you not take the time to do so.

Rewrite the following using rational and negative exponents:

$\sqrt[5]{x^3} + \sqrt[5]{2x}$

$\sqrt{x+1}$

$\frac{1}{\sqrt{x+1}}$

$\frac{1}{\sqrt{x}} - \frac{2}{x}$

$\frac{1}{4x^3} + \frac{\sqrt[4]{x^3}}{2}$

$\frac{1}{4\sqrt{x}} - 2\sqrt{x+1}$

Rewrite each expressions in radical form and positive exponents:

$x^{-1/2} - x^{3/2}$

$\frac{1}{2}x^{-1/2} + x^{-1}$

$3x^{-1/2}$

$(x+4)^{-1/2}$

$x^{-2} + x^{1/2}$

$2x^{-2} + \frac{3}{2}x^{-1}$

EXPONENTIAL FUNCTIONS AND LOGARITHMS

Solve the following equations. Remember that $e^{0} = 1$ *and* $ln\,1 = 0$

$e^{x} + 1 = 2$	$3e^{x} + 5 = 8$	$e^{2x} = 1$
$ln\,x = 0$	$3 - ln\,x = 3$	$ln(3x) = 0$
$x^{2} - 3x = 0$	$e^{x} + xe^{x} = 0$	$e^{2x} - e^{x} = 0$

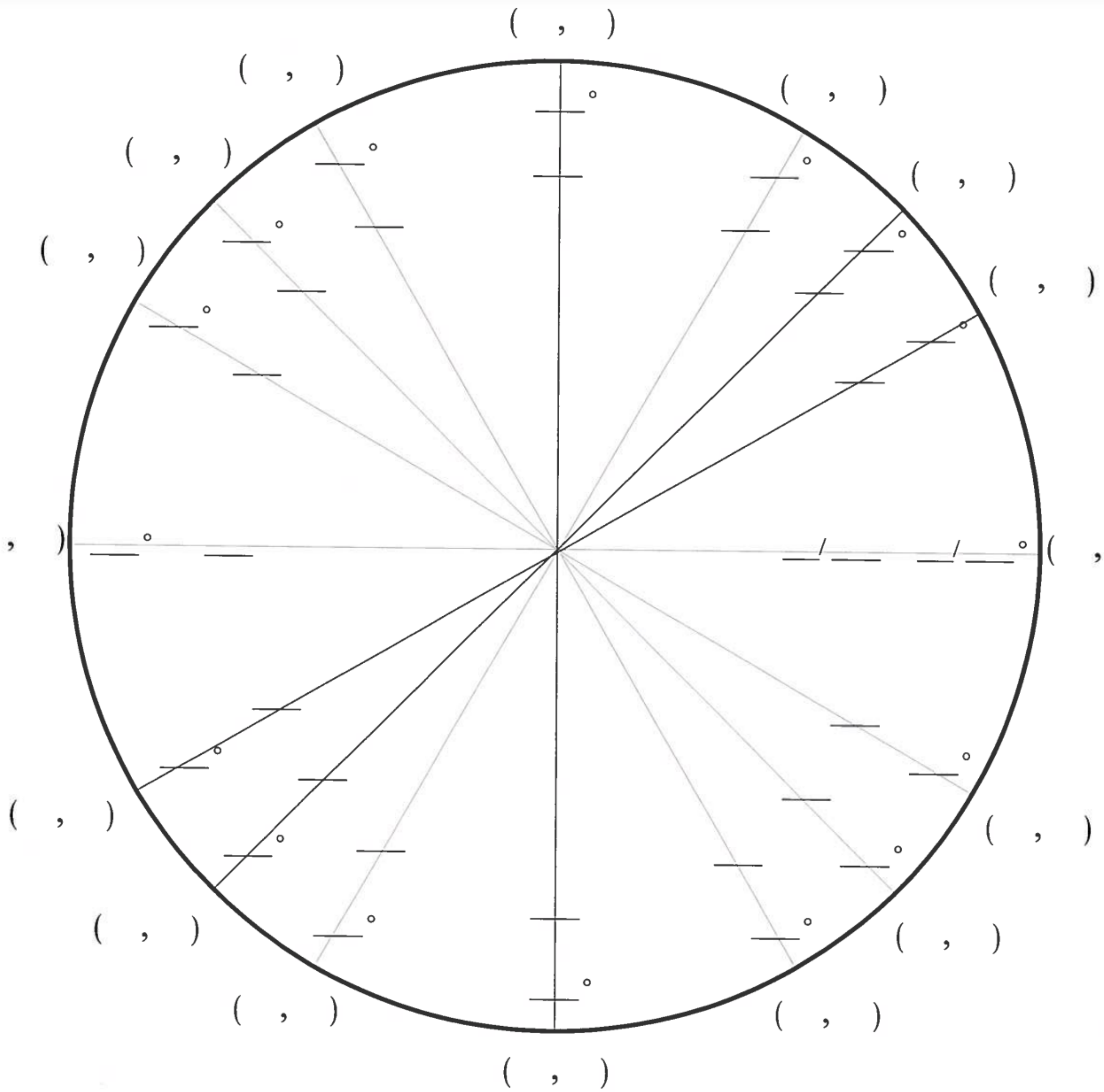
Simplify the following

$e^{ln\,x}$	$e^{1+ln\,x}$	$ln\,1$	$ln\,e^{7}$
$log_{3}\frac{1}{3}$	$log_{1/2}8$	$ln\,\frac{1}{2}$	$27^{2/3}$

TRIGONOMETRY

Knowing your Unit Circle is a MUST. Not only should students be able to fill out a unit circle, but they should be able to pull answers from any part of the unit circle quickly and correctly.

Complete the unit circle:



Without a calculator, evaluate the trig functions exactly.

$\sin\left(\frac{\pi}{2}\right)=$	$\cos\left(\frac{\pi}{3}\right)=$	$\tan\left(\frac{\pi}{4}\right)=$	$\sin(\pi)=$
$\sin\left(\frac{\pi}{6}\right)=$	$\cos(2\pi)=$	$\sin\left(\frac{\pi}{3}\right)=$	$\cos\left(\frac{2\pi}{3}\right)=$
$\tan\left(\frac{\pi}{2}\right)=$	$\sin\left(\frac{\pi}{4}\right)=$	$\tan(\pi)=$	$\cos\left(\frac{3\pi}{4}\right)=$
$\tan(2\pi)=$	$\tan\left(\frac{3\pi}{4}\right)=$	$\cos\left(\frac{3\pi}{2}\right)=$	$\cos\left(\frac{\pi}{6}\right)=$
$\cos\left(\frac{\pi}{4}\right)=$	$\cos\left(\frac{\pi}{2}\right)=$	$\cos(\pi)=$	$\sin(2\pi)=$
$\sin\left(\frac{2\pi}{3}\right)=$	$\sin\left(\frac{3\pi}{4}\right)=$	$\sin\left(\frac{3\pi}{2}\right)=$	$\tan\left(\frac{3\pi}{2}\right)=$

TRIG IDENTITIES

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier. The following chart gives the major trig identities that you should know. To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

Fundamental Trig Identities

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$

Sum Identities

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Double Angle Identities

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Verify the following identities:

$$(1 + sin\,x)(1 - sin\,x) = cos^2 x$$

$$sec^2 x + 3 = tan^2 x + 4$$

$$\frac{1 - sec\,x}{1 - cos\,x} = -\,sec\,x$$

$$\frac{1}{1 + tan\,x} + \frac{1}{1 + cot\,x} = 1$$

$$\frac{cos\,x - cos\,y}{sin\,x + sin\,y} + \frac{sin\,x - sin\,y}{cos\,x + cos\,y} = 0$$

$$\frac{sin^3 x + cos^3 x}{sin\,x + cos\,x} = 1 - sin\,x\,cos\,x$$